

Table of Contents

Dedication	5
Foreword	7
Preface	9
Synopsis	11
Table of Contents	13
I Lyran Algebra Lessons Book 1: Fundamentals of Polynomials	17
1 Polynomial Foundations	19
1.1 Polynomial Foundations and Standard Transformations	20
1.2 Casus Irreducibilis and Cubic Roots	41
1.3 Roots, Factorization, and Structural Consequences	44
1.3.1 Rational Roots of Integer Polynomials	44
1.4 Viète's Formulas and Symmetric-Sum Techniques	56
1.5 Quartic and Quintic Polynomials	63
2 Algebraic and Polynomial Expressions	71
2.1 Functional Expressions: Substitution, Composition, and Invariance Ideas	72
2.1.1 Solved Examples: Functional Expressions	73
2.1.2 Exercises: Functional Expressions	77
2.2 Polynomial Expressions: Degree, Factoring Patterns, Simplification Strategies	81
2.2.1 Solved Examples: Polynomial Expressions	81
2.2.2 Exercises: Polynomial Expressions	83
3 Polynomial Factorization Methods	91
3.1 Univariate Factorization	92
3.1.1 Solved Examples: Univariate Factorization	92
3.1.2 Exercises: Univariate Factorization	95
3.2 Bivariate Factorization	97
3.2.1 Solved Examples: Bivariate Factorization	97
3.2.2 Exercises: Bivariate Factorization	100
3.3 Trivariate Factorization	102
3.3.1 Solved Examples: Trivariate Factorization	102
3.3.2 Exercises: Trivariate Factorization	109
3.4 Multivariate Factorization	112
3.4.1 Solved Examples: Multivariate Factorization	112
3.4.2 Exercises: Multivariate Factorization	115
4 Polynomial Division and the Euclidean Algorithm	121
4.1 The Division Algorithm	122

4.2	Remainders, Evaluations, and Polynomial Congruences	141
4.3	Polynomial GCDs and the Euclidean Algorithm	152
5	Advanced Polynomial Theory and Methods	159
5.1	Essential Advanced Polynomial Theorems	160
5.1.1	Definitions and Structural Theorems	160
5.1.2	Solved Examples: Structural Theorems	163
5.2	Advanced Polynomial Theorems and Applications	166
5.2.1	Descartes' Rule of Signs and Number of Negative Roots	166
5.2.2	Resultant and Discriminant of Polynomials	170
5.2.3	Lagrange's Interpolation and Finite Differences	174
5.3	Symmetric Sums in Olympiads: Identities, Inequalities, and Reconstruction	184
5.3.1	Solved Examples: Symmetric Sums	184
5.3.2	Exercises: Symmetric Sums	195
5.4	Advanced Polynomial Problems	197
5.4.1	Solved Examples: Advanced Polynomials	197
5.4.2	Exercises: Advanced Polynomials	206
6	Complex Numbers and Polynomial Geometry	209
6.1	The Complex Plane	210
6.2	Modulus, Conjugation, and Rotation	211
6.3	Complex Number Problems	215
6.3.1	Solved Examples: Complex Numbers	215
6.3.2	Exercises: Complex Numbers	234
7	Algebraic Systems and Equation Methods	239
7.1	Essential Tools for Algebraic Systems	240
7.1.1	Introduction to Algebraic Systems	240
7.1.2	Tools for Solving Algebraic Systems	245
7.2	Polynomial Equations: Standard Forms, Transformations, Solving Strategies	247
7.2.1	Solved Examples: Polynomial Equations	247
7.2.2	Exercises: Polynomial Equations	260
7.3	Trigonometric Tricks for Algebraic Systems	263
7.3.1	Solved Examples: Trigonometric Substitutions	263
7.3.2	Exercises: Trigonometric Substitutions	272
7.4	Irrational Equations: Radicals, Squaring Strategies, Extraneous-Root Control	276
7.4.1	Solved Examples: Irrational Equations	276
7.4.2	Exercises: Irrational Equations	285
7.5	Reciprocal & Palindromic Equations: Inversion Symmetry & Key Substitutions	290
7.5.1	Solved Examples: Reciprocal Equations	290
7.5.2	Exercises: Reciprocal Equations	302
7.6	Piecewise & Discrete Equations: Floors, Ceilings, Absolute Values, Casework	305
7.6.1	Solved Examples: Floors, Ceilings, and Absolute Values	305
7.6.2	Exercises: Floors, Ceilings, and Absolute Values	318
7.7	Arithmetic of Polynomial Roots: Power Sums, Transformations, Root Relations	322
7.7.1	Further Problems on Polynomial Roots	322
7.7.2	Power Sums of Roots: Newton's Identities & Linear Recurrences	345
7.7.3	Constructing Polynomials from Roots: Reverse Viète & Transformations	352
7.7.4	Common Roots: Elimination, Resultants, Parameter Conditions	375
7.8	Mixed Systems of Equations: Symmetry, Substitution, Olympiad Casework	389

7.8.1	Solved Examples: Mixed Systems	390
7.8.2	Exercises: Mixed Systems	407
8	Arithmetic Methods for Polynomial Problems	415
8.1	Divisibility, Orders, and Lifting	417
8.1.1	Arithmetic Definitions and Theorems.....	417
8.1.2	Solved Examples: Arithmetic Methods	421
8.2	Polynomials Modulo an Integer	428
8.2.1	Modular Polynomial Theorems	428
8.2.2	Lagrange's Lemma and The Most Useful Number Theoretic Polynomial	430
8.2.3	Legendre's Formula, Digit-sum Form, and p -adic Power Sums	435
8.2.4	Lucas and Kummer in a Freshman's Dream	437
8.2.5	Advanced Analytic Theorems on Polynomials	441
8.3	Number-Theoretic Polynomial Problems.....	444
8.3.1	Solved Examples: Number-Theoretic Polynomials	444
8.3.2	Exercises: Number-Theoretic Polynomials	454
9	Selected Polynomial Problems	463
9.1	Integer, Rational, or Number-Theoretic Constraints	464
9.2	Irreducibility, Factorization, and Decomposition	480
9.3	One-Variable Functional Equations and Iteration	489
9.4	Trigonometric, Chebyshev, and Symmetry Problems	509
9.5	Roots, Coefficients, and Inequalities	522
9.6	Summation and Finite Difference Identities.....	536
9.7	Multivariate Polynomials and Homogeneity.....	545
9.8	Algebraic Numbers, Conjugation, and Radical Identities	554
10	Classical and Contest Polynomial Problems	561
10.1	Solved Examples: Classical Problems.....	562
10.2	Recorded Contest Problems: Foundations.....	601
10.3	Recorded Contest Problems: Advanced Methods.....	601
10.4	Recorded Contest Problems: Synthesis	624
11	Polynomial Synthesis: Equations and Complex Methods	649
11.1	Mixed-Method Mock Contest	650
11.2	Past Competition Problems.....	669
11.3	Original Synthesis Problems.....	677
	Bibliography	683
	Physical Resources.....	683
	Online Resources	684
	Acronyms.....	689
	List of Problems in Lessons Book	691
	Indexes	705
	Index of Concepts.....	705
	Index of Methods	709
	Index of Theorems	713
	Index of People	715
	Index of Tools & Problem Types	717
	Index of Sources	721